Delay Induced Excitability

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We analyse the stochastic dynamics of a bistable system under the influence of time-delayed feedback. Assuming an asymmetric potential, we show the existence of a regime in which the systems dynamic displays excitability by calculating the relevant residence time distributions and correlation times. Experimentally we then observe this behaviour in the polarization dynamics of a vertical cavity surface emitting laser with opto-electronic feedback. Extending these observations to two-dimensional systems with dispersive coupling we finally show numerically that delay induced excitability can lead to the appearance of propagating wave-fronts and spirals.

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The influence of time delayed feedback on the dynamics of otherwise Markovian processes with noise is currently a very active topic of research, as memory effects have been found to be important in many systems in areas ranging from biology to medicine and electronics [1]. Even though many of the general properties of systems with memory are still unexplored, various methods for engineering of the delay to induce instabilities or stabilize complex dynamics have already been suggested [2].

As a generic model to study the effects of delay on a bi-stable stochastic system one can consider a particle trapped in a double well potential. In the absence of feedback the particle makes random transitions between the two minima of the potential and two consecutive jumps can be considered as being independed of each other. When feedback is present, however, correlations are introduced into the system and the dynamics becomes inherently non-Markovian. Let us consider the following standard stochastic differential equation

$$\dot{x} = -\frac{dU}{dx} + \epsilon x(t-\tau) + \eta(t), \qquad (1)$$

which describes an over-damped motion in a double well potential $U(x)$ under the influence of memory. The length of the delay interval is τ and ϵ is the feedback strength. Here $\eta(t)$ describes Gaussian white noise with $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = 2D\delta(t-t')$. In the following we will denote the delayed state as $x_\tau = x(t-\tau)$.

Without delay ($\epsilon = 0$), the random switching between the two minima of the potential, $x_{l,r}$, can be very well described by Kramers' theory [3]. The residence time distribution (RTD) $P_{l,r}(T)$ in each well is then given by

$$P_{l,r}(T) = \gamma_{l,r}\; e^{-\gamma_{l,r}T}, \qquad (2)$$

where the inverse of the switching rate is the so-called Kramers' time, $T_{l,r} = 1/\gamma_{l,r}$. It describes the average time between two transitions

$$T_{l,r} = \frac{2\pi}{\sqrt{U''(x_{l,r})U''(x_0)}} \exp\left[\Delta U_{l,r}/D\right]. \qquad (3)$$

Here x_0 is the position of the local maximum between the two minima $x_{l,r}$ and $\Delta U_{l,r} = U(x_0) - U(x_{l,r})$ is the potential barrier for each well.

The statistical properties of eq. (1) for $\epsilon \neq 0$ have recently been analysed in the case of the generic symmetric potential $U(x) = -x^2/2 + x^4/4$. Noting that the potential can be rewritten to include the delay term, $V(x) = U(x) + \epsilon x\, x_\tau$, it can immediately be seen that the potential barrier, and therefore the escape rates, become dependent on the state at the earlier time $t-\tau$ [4]. Hence, the continuous system of eq. (1) can be approximated by a discrete two-state model, $r(t) = \pm 1$, with transition rates that depend on the delayed state [5]. These rates depend on the relative value of $r(t)$ and $r(t-\tau)$ and are defined as p_1 if $r(t) = r(t-\tau)$ and p_2 if $r(t) \neq r(t-\tau)$ for a symmetric potential. Tsimring and Pikovsky have recently derived the power spectrum for this model and shown the existence of coherence resonance [5]. Other works have calculated and measured the RTD [4, 6, 7].

Here we show that the presence of an asymmetry in the potential can lead to the appearance of excitability [8]. In an excitable system a perturbation above a certain threshold transfers an otherwise stable steady-state (the resting state) into an excited state (the firing state). The system then goes through a well defined refractory cycle, before returning to its initial state. During the refractory

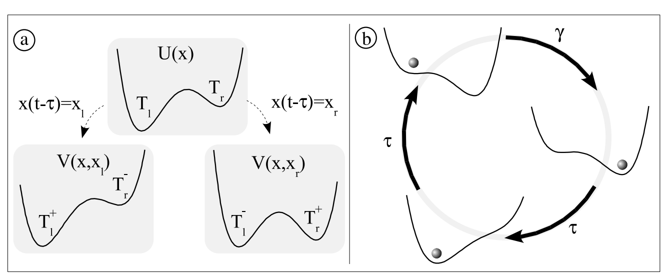

FIG. 1: (a) Schematic definition of Kramers' residence times for an asymmetric potential without (upper graph) and with (lower graphs) negative feedback. (b) Schematic of the potentials during one excitable cycle.

cycle it is impervious to any external perturbation and cannot 'fire' a second time. Systems following this kind of dynamics are widespread in biology, chemistry and many other areas and there are several mathematical models existing to describe such dynamics [8]. It is worth pointing out, that for noise driven excitable systems different manifestations of a higher degree of order at a finite noise level have been discovered, for example coherence resonance [9] and wavefronts and even self-ordered patterns like spirals in spatially extended systems [10] .

For the asymmetric potential the presence of feedback leads to a dependence of the amplitude of each potential barrier on the position of the particle at the time $t-\tau$ and the dynamics has to be described by four characteristic times $T^{\pm}_{l,r}$. The times T^{+}_{l} and T^{-}_{l} are the mean escape times from the left well when the particle was in the left or right well at $t-\tau$, respectively, and T^{-}_{r} and T^{+}_{r} are the mean escape times from the right well if the particle was in the left or right well at $t-\tau$, respectively (see Fig. 1a).

Even though all transitions are inherently stochastic, we show below that for negative feedback ($\epsilon < 0$) a parameter region exists, in which the systems displays excitable behaviour. To illustrate this let us consider the case of an asymmetric potential with $T_r \ll T_l$ (for $\epsilon = 0$) such that the minimum corresponding to $x < 0$ exists for any value of the feedback, x_τ, but the minimum for $x > 0$ only exists for $x_\tau < 0$ (see Fig. 1b). Let us also assume that the particle has been in the left well for $t < 0$ and it is driven by noise into the right well at $t = 0$. For small noise amplitudes ($T^{-}_{r} \gg \tau$) the particle will almost certainly remain in the right well until the value of the feedback term changes at $t = \tau$. The potential minimum on the right hand side then disappears and the particle will move toward the remaining minimum at $x < 0$. After a further time interval τ, the potential will switch to its initial bi-stable shape and the above process can start over again. Such a cycle is characteristic of an excitable system. The initial state is stable under small perturbations but large perturbations can induce a jump. The firing corresponds to the transition of the particle into the right well and the refractory time consists of the two subsequent periods of duration τ before the initial state is established again. During the first one the refractory time the particle resides in the well $x > 0$, from which transitions are unlikely and during the second one only one minimum is present and transitions cannot occur.

To show that the dynamics described above can indeed be observed from eq. (1) we have numerically calculated the RTDs in the excitable regime (see Fig. 2). The potential was chosen to be

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{3}\kappa x^3, \tag{4}$$

with $\kappa = 0.3$ and $\epsilon = 1/3$. Assuming that the particle has made a transition into the right well (the system has fired) at the time $t = 0$, one can clearly see from Fig. 2 that the transition probability back into the left well (dashed line) is effectively zero in the interval $[0, \tau]$ The spike appearing for $T > \tau$ then indicates the non-stochastic roll-over to the left well once the feedback has changed and the potential possesses only the single minimum on the left hand side. From there no further transition is possible until at a time τ later the potential changes back to its bistable shape (full line in Fig. 2). After that the distribution for $t > \tau$ follows Kramers' law. Fig. 2 clearly shows the existence of a refractory time that forbids firing twice in an interval of 2τ.

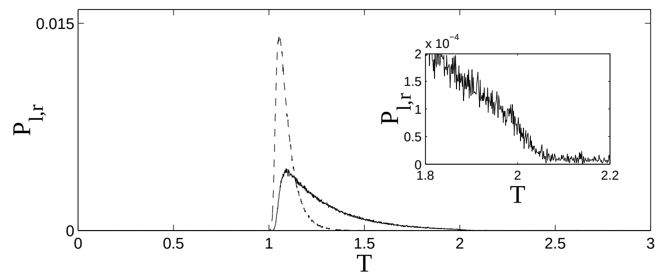

FIG. 2: Numerical RTDs for the left (full line) and right well (dashed line). The delay interval is $\tau = 1$. The inset shows the change in slope at $T = 2$ for particles in the left well, which is due to a higher order effect in the feedback strength.

Let us compare our continuous system to a discrete model. Although the simplest discrete model to describe excitability can be constructed using a two-state system, it was recently shown, that networks of excitable units display a wider spectrum of dynamical effects when they are approximated by three state models [11, 12]. In these models only the transition from the resting into the firing state is stochastic, while the other two transitions are deterministic with fixed waiting times. It is clear from the discussion above that such a model is very close to our bistable system with feedback. Even though all transitions in the potential are at all times inherently stochastic, the two intervals during the cycle in which the transition probability is very low can be well approximated by a waiting time of length $T_w = \tau$ (see Fig. 2).

To compare the power spectra we will assume that the stochastic output of the discrete model is described by a function that has a value 1 if the system is in the firing state and is zero otherwise. The spectrum can then be calculated analytically [18]

$$S(\omega) = \frac{2 - 2\cos(\omega\tau)}{(\gamma + 2\gamma^2\tau)(1 - \cos(2\omega\tau) + \frac{\omega}{\gamma}\sin(2\omega\tau) + \frac{\omega^2}{2\gamma^2})} \tag{5}$$

Fig. 3 shows the power spectrum given by eq. (5) (upper graph) and calculated from numerical simulations of eq. (1) with the potential of eq. (4) (lower graph). Both curves are similar in shape and show their maxima close to odd multiples of π and minima at even multiples of

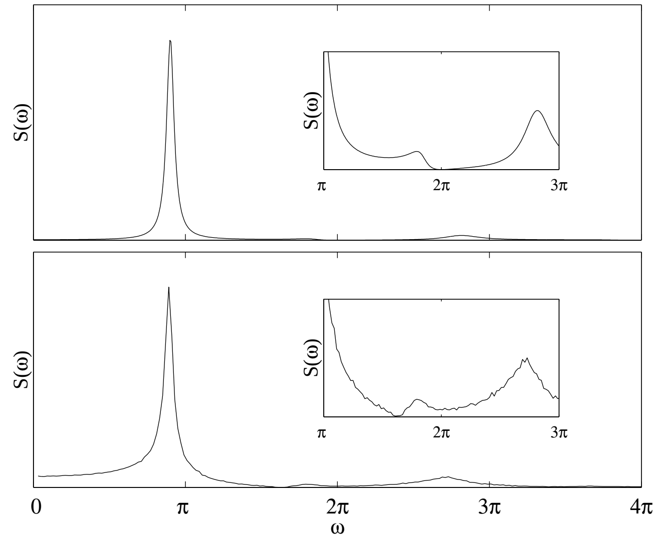

FIG. 3: Power spectrum of the discrete (upper) and the continuous model (lower). Here $\tau = 1$, $\epsilon = 0.33$, $\kappa = 0.3$.

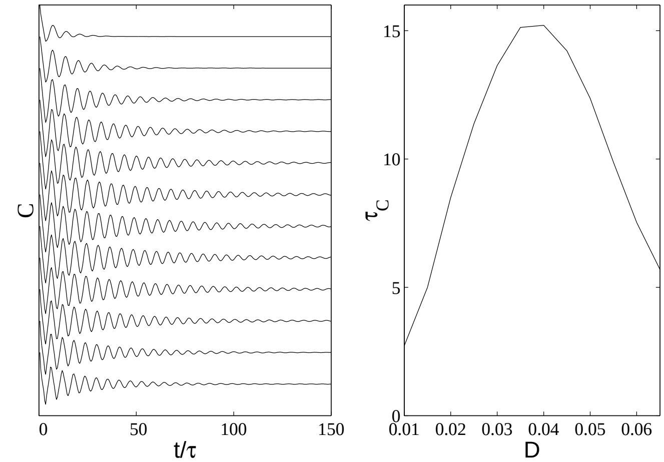

FIG. 4: Left: autocorrelation function of the system with increasing noise. For the uppermost curve $D = 0.01$ which increases downwards in steps of $\Delta D = 0.005$. Right: correlation times. The maximum is a clear sign of ordering.

π. However, it must be noted that for the discrete model the coherence is an increasing function with noise [11, 12], while for the continuous model it is maximized for a finite strength of the noise. To demonstrate this resonance we have calculated the autocorrelation function and the associated correlation times, τ_C, as a function of noise strength. The correlation time is defined as the integral over the square of the autocorrelation function [9]

$$T_c = \int_0^\infty \left[\frac{\langle \tilde{x}(t)\tilde{x}(t+t')\rangle}{\langle \tilde{x}(t)^2\rangle} \right]^2 dt', \qquad \tilde{x} = x - \langle x\rangle \ . \quad (6)$$

The left hand side of Fig. 4 shows the autocorrelation functions with noise increasing from the top to the bottom graph. It is clearly visible that the correlations are strongest for a finite noise level, which is confirmed in the plot on the right hand side, where the τ_C show a clear maximum for a noise amplitude of $D \sim 0.04$. This behaviour is closely related to the appearance of a phase transition from excitable to oscillatory media [13, 14].

EXPERIMENTAL RESULTS

Excitability has been observed in the intensity of semiconductor lasers with optical feedback [16]. Here we experimentally analyse the behaviour of the polarisation dynamics of a Vertical Cavity Surface Emitting Laser (VCSEL) with opto-electronic feedback as described in [4, 7]. The light emitted by a VCSEL is usually linearly polarised but, as the injection current increases, the emission axis will switch between the two orthogonal states. Around this switching point one can find a current range in which the light polarisation randomly switches between the two axes. Previous experiments have been designed using these lasers to study Kramers' law [15], stochastic resonance [17] and the properties of noisy time delay dynamical systems [4, 7]. It is worthwhile noticing that the shape of the potential depends on the bias current applied to the laser. At the center of the bistable region both polarisations occur with the same probability and the associated potential is symmetric. The potential becomes asymmetric as the injection current is varied toward the boundary of the the bistable region and our laser was operated in this region in order to observe delay-induced excitability.

Although spontaneous polarisation switching was observed without the addition of external noise, a noise generator was added in order to observe coherence resonance. Fig. 5 (left) shows the measured RTDs for the lowest noise in the excitable regime where P_x, P_y refer to the orthogonal linear polarisation states. The important feature is the absence of switching events for times less than τ. Additionally, both the stochastic nature of the firing transition and deterministic nature of the refractory cycle can be seen in the spread of switching times being much less for P_y than P_x. Coherence resonance for this operating point is shown in Fig. 5 (right) and qualitatively reproduces the predicted numerical behaviour.

TWO-DIMENSIONAL SYSTEMS

The ability to sustain spatio-temporal patterns in systems of coupled stochastic units is an important property of excitable systems [10]. It was shown that wave-propagation and spiral patterns [19] are common in excitable systems and here we confirm their existence for spatially coupled elements as described by eq. (1). We consider a situations where the elements are coupled via

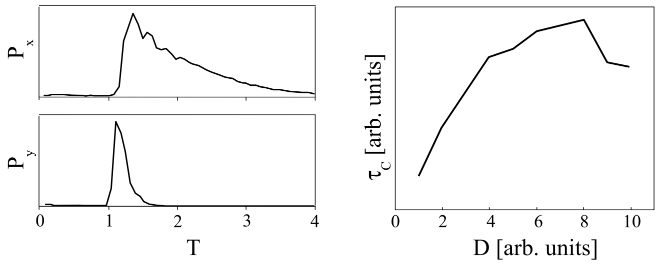

FIG. 5: Left: Experimental RTDs for the x and y polarization of the VCSEL. Excitability is clearly visible from absence of residence times shorter than τ. Right: Correlation times of the VCSEL output signal for increasing noise levels.

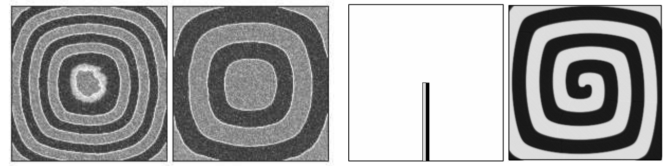

FIG. 6: Left: Two dimensional wave-surfaces. The delay time in the left panel is twice the value of the delay time in the right panel. Right: Spiral pattern emerging from the initial condition (left panel) as described in the text. Black corresponds to the system in the firing state, grey to the refractory state and white to the normal state. The parameters here are $\tau = 80$, $\epsilon = 0.343$, $\kappa = 0.07$ and $D = 0$.

a diffusion term

$$\frac{\partial s}{\partial t} = s - \kappa s^2 - s^3 - \epsilon s_\tau + \sqrt{2D}\xi + \nabla^2 s\ . \qquad (7)$$

Let us choose each element of the system to be initially in the resting state. At some point the noise will trigger one of the elements to fire and the coupling to neighboring elements will lead to short time correlations: in order to reduce the 'kinetic energy' between two neighbouring elements they will also undergo a transition into the firing state. After the refractory time the elements will transfer back in the initial state, marking the end of the wave-front. However, due to higher order effects in the feedback strength [7], the states at the end of the refractory time do not have the same Kramers' time as the initial ones. In fact, for negative feedback the new Kramers' time is much smaller and these elements are more prone to being excited again. This leads to the continuous creation of wave surfaces with wavelength τ. In Fig. 6 (left) we show two examples of such wave surfaces, where the delay time of the system on the right is twice as long as in the system shown on the left hand side. The two dominating colors represent the two refractory states for $s > 0$ and $s < 0$. It is clear that these ordered waves can only be sustained at an optimal noise level. Too little noise will not lead to any excitations whereas too much noise will lead to fragmentation of the wavefronts.

Instead of using noise to trigger the spatial ordering, one can also choose appropriate initial conditions. Here we demonstrate the appearance of spiral patterns by preparing a system with one area where the elements have just fired after having been in the resting state for $[-\tau, 0[$. A neighboring area has just made the transition back into the resting state, while having been in the refractory state for $[-\tau, 0[$. The rest of the system is in the resting state and has been there for $[-\tau, 0]$ (see Fig. 6 (right)). Integrating from these intial conditions leads to the evolution of a spiral as shown on the right hand side in Fig. 6.

In summary we have shown that including delay into bistable systems can lead to excitable behaviour. We have theoretically investigated the model of an asymmetric double well potentials and found clear signatures of excitability in the RTD. We have also compared the continuous to a discrete model for excitability and found good agreement of the power spectra. We have calculated the coherence times and shown the existence of coherence resonance. Experimentally we have demonstrated the above behaviour using a VCSEL in the bistable regime.For spatially extended coupled systems we have demonstrated the appearance of spatio-temporal order in the form of waves and spirals.

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